

Volume Preserving Multidimensional Integrable Systems and Nambu–Poisson Geometry

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Abstract

In this paper we study generalized classes of volume preserving multidimensional integrable systems via Nambu–Poisson mechanics. These integrable systems belong to the same class of dispersionless KP type equation. Hence they bear a close resemblance to the self dual Einstein equation. All these dispersionless KP and dToda type equations can be studied via twistor geometry, by using the method of Gindikin’s pencil of two forms. Following this approach we study the twistor construction of our volume preserving systems.

Dedicated to the memory of Dr. B C Guha

1 Introduction

In this article we study volume preserving diffeomorphic integrable hierarchy of three flows [1]. This is different from the usual two flows cases, and this can be studied via Nambu–Poisson geometry. It is already known that a group of volume preserving diffeomorphisms in three dimension plays a key role in an Einstein–Maxwell theory where the Weyl curvature is self-dual and the Maxwell tensor is algebraically anti self-dual. Later Takasaki [2] explicitly showed how volume preserving diffeomorphisms arises in integrable deformations of self-dual gravity.

Nambu mechanics is a generalization of classical Hamiltonian mechanics, introduced by Yoichiro Nambu [3]. At the beginning he wanted to formulate a statistical mechanics on \mathbb{R}^3 , emphasizing that the only feature of Hamiltonian mechanics one should preserve is the Liouville theorem. He considered the following equations of motion

$$\frac{d\mathbf{r}}{dt} = \nabla u(\mathbf{r}) \wedge \nabla v(\mathbf{r}), \quad \mathbf{r} = (x, y, z) \in \mathbb{R},$$

where x, y, z are dynamical variables and u, v are two functions of \mathbf{r} . Then Liouville theorem follows from the identity

$$\nabla \cdot (\nabla u(\mathbf{r}) \wedge \nabla v(\mathbf{r})) = 0.$$

He further observed from above equation that the evolution of a function on \mathbb{R}^3 is given by

$$\frac{df}{dt} = \frac{\partial(f, u, v)}{\partial(x, y, z)},$$

where the right hand side is the Jacobian of the mapping $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by

$$(x, y, z) \longrightarrow (f, u, v).$$

The right hand side can be interpreted as a generalized Poisson bracket. Hence the binary operation of Poisson bracket of Hamiltonian mechanics is generalized to n -ary operation in Nambu mechanics. Recently Takhtajan [4, 5] has formulated its basic principles in an invariant geometrical form similar to that of Hamiltonian mechanics.

In this paper, we shall use Nambu mechanics to study generalized volume preserving diffeomorphic integrable hierarchy. These classes of integrable systems are closely related to the self dual Einstein equation, dispersionless KP equations etc. In fact we obtain a higher dimensional analogue of all these systems. It turns out that all these systems can be written in the following form:

$$d\Omega^{(n)} = 0, \quad \Omega^{(n)} \wedge \Omega^{(n)} = 0.$$

For $n = 2$ we obtain all the self dual Einstein and dKP type equations.

Hence we obtain a common structure behind all these integrable system, so there is a consistent and coherent way to describe all these systems. Here we unify all these classes of integrable systems by Gindikin's pencil or bundle of forms and Riemann–Hilbert problem (twistor description). Gindikin introduced these technique to study the geometry of the solution of self dual Einstein equations. Later Takasaki–Takebe [6, 7] applied it to dispersionless KP and Toda equations.

This paper is organized as follows. In Section 2 we present a tacit introduction of Nambu–Poisson geometry. In Section 3 we construct our volume preserving integrable systems via Nambu–Poisson geometry. If one carefully analyse these set of equations, then one must admit that they bear a close resemblance with the *volume preserving KP equation*, so far nobody knows about this equation. It is known that area preserving KP hierarchy (= dispersionless KP hierarchy) plays an important role in topological minimal models (Landau–Ginzburg description of the A-type minimal models). So we expect volume preserving KP hierarchy may play a big role in low dimensional quantum field theories. Section 4 is dedicated to the twistor construction of these systems.

Notations

1. \mathcal{M} : Nambu-Poisson manifold.
2. η : Nambu-Poisson polyvector (tensor).
3. J_η : Bundle map associated to η .
4. $\mathcal{X}_{f_1 \dots f_{n-1}}$: Hamiltonian vector field associated to η .
5. L, M, N : Formal Laurent series, Lax operators involving volume preserving systems.

6. \mathcal{L} : Lax operator for dispersionless KP hierarchy.
7. \mathcal{K} : Orlov operator in dispersionless KP.
8. Ω : two form in dispersionless KP.
9. $\Omega^{(3)}$: three form in volume preserving integrable systems.
10. \mathcal{T} : curved twistor space.
11. (P, Q) or (P', Q') : Pair of Darboux coordinates.
12. e^{ij} : one forms or tetrad.

2 Nambu–Poisson manifolds

The modern concept of Nambu–Poisson structure was proposed by Takhtajan in 1994 in order to find an axiomatic formulation for the n -bracket operation. Let \mathcal{M} denote a smooth n -dimensional manifold and $C^\infty(\mathcal{M})$ the algebra of infinitely differentiable real valued functions on \mathcal{M} . A manifold \mathcal{M} is called a Nambu–Poisson manifold if there exists a \mathbb{R} -multi-linear map

$$\{ \cdot, \dots, \cdot \} : [C^\infty(\mathcal{M})]^{\otimes n} \rightarrow C^\infty(\mathcal{M}).$$

This is called Nambu–Poisson bracket of order $n \forall f_1, f_2, \dots, f_{2n-1} \in C^\infty(\mathcal{M})$. This bracket satisfies

$$\{f_1, \dots, f_n\} = (-1)^{\epsilon(\sigma)} \{f_{\sigma(1)}, \dots, f_{\sigma(n)}\}, \quad (2.1)$$

$$\{f_1 f_2, f_3, \dots, f_{n+1}\} = f_1 \{f_2, f_3, \dots, f_{n+1}\} + \{f_1, f_3, \dots, f_{n+1}\} f_2, \quad (2.2)$$

and

$$\begin{aligned} & \{\{f_1, \dots, f_{n-1}, f_n\}, f_{n+1}, \dots, f_{2n-1}\} \\ & + \{f_n, \{f_1, \dots, f_{n-1}, f_{n+1}\}, f_{n+2}, \dots, f_{2n-1}\} + \dots \\ & + \{f_n, \dots, f_{2n-2}, \{f_1, \dots, f_{n-1}, f_{2n-1}\}\} \\ & = \{f_1, \dots, f_{n-1}, \{f_n, \dots, f_{2n-1}\}\}, \end{aligned} \quad (2.3)$$

where $\sigma \in S_n$ — the symmetric group of n elements — and $\epsilon(\sigma)$ is its parity. Equations (2.1) and (2.2) are the standard skew-symmetry and derivation properties found for the ordinary ($n = 2$) Poisson bracket, whereas (2.3) is a generalization of the Jacobi identity and was called in [2] the fundamental identity. When $n = 3$ this fundamental identity reduces to

$$\begin{aligned} & \{\{f_1, f_2, f_3\}, f_4, f_5\} + \{f_3, \{f_1, f_2, f_4\}, f_5\} \\ & + \{f_3, f_4, \{f_1, f_2, f_5\}\} = \{f_1, f_2, \{f_3, f_4, f_5\}\}. \end{aligned}$$

2.1 Hamiltonian geometry

The Nambu–Poisson bracket is geometrically realized by the Nambu–Poisson polyvector $\eta \in \Gamma(\wedge^n TM)$, a section of $\wedge^n TM$, such that

$$\{f_1, \dots, f_n\} = \eta(df_1, \dots, df_n),$$

in local coordinates (x_1, \dots, x_n) this is given by

$$\eta = \eta_{i_1 \dots i_n}(x) \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_n}},$$

where summation over repeated indices are assumed.

The structure is regular if $\eta \neq 0$. The canonical example of a Nambu–Poisson structure of order $n \geq 2$ is the one induced by a volume form on an oriented manifold

$$\{f_1, \dots, f_n\} = \eta(df_1, \dots, df_n).$$

The Nambu–Poisson polyvector defines a bundle map

$$J_\eta : \underbrace{T^*\mathcal{M} \times \dots \times T^*\mathcal{M}}_{n-1} \longrightarrow T\mathcal{M}$$

given by

$$\langle \beta, J_\eta(\alpha_1, \dots, \alpha_{n-1}) \rangle = \eta(\alpha_1, \dots, \alpha_{n-1}, \beta),$$

where $\beta, \alpha_i \in \Omega^1(M)$.

The Hamiltonian vector field is defined by

$$\mathcal{X}_{f_1, \dots, f_{n-1}} = J_\eta(df_1, \dots, df_{n-1}).$$

This is a Hamiltonian vector field of $(n-1)$ functions. Thus, we can express the fundamental identity as

$$L_{\mathcal{X}_{f_1, \dots, f_{n-1}}} \eta(dg_1 \cdots dg_n) = 0,$$

where L is the Lie derivative. This shows that the Hamiltonian vector fields are the infinitesimal automorphism of the Nambu–Poisson tensor.

The polyvector field $\eta \in \Gamma(\wedge^n TM)$ defines an n -ary Poisson bracket if and only if either n is even and the Schouten bracket satisfies $[\eta, \eta] = 0$, or n is odd if η satisfies the conditions (see for detail [8])

$$i_\alpha(\eta) \wedge i_\beta(\eta) = 0, \quad \forall \alpha, \beta \in T^*\mathcal{M}, \quad \sum_{j=1}^n (i_{dx_j} \eta) \wedge \left(L_{\frac{\partial}{\partial x_j}} \eta \right) = 0.$$

In this framework we can introduce the bihamiltonian structure also. A bihamiltonian system is prescribed by specifying two sets of Hamiltonian function $H = H_1, \dots, H_n$ and $h = h_1, \dots, h_n$, where $H_i, h_i \in C^\infty(\mathcal{M})$,

$$\mathcal{X} = J_\eta^0(dh_1, \dots, dh_n) = J_\eta^1(dH_1, \dots, dH_n),$$

where J_η^i s are the bundle maps.

One more interesting algebraic structure pops up in the Nambu–Poisson geometry — this is called Leibniz algebra. The Nambu–Poisson polyvector η induces a homomorphism of vector bundles

$$J_\eta^k : \wedge^k(T^*\mathcal{M}) \longrightarrow \wedge^{n-k}(TM),$$

such that $J_\eta^k(\beta) = i_\beta \eta$ for $\beta \in \wedge^k T_x^* \mathcal{M}$. All these higher covectors form Leibniz algebra. It was shown by Ibanez et. al. [9] that the bracket of $(n-1)$ forms on \mathcal{M} is the \mathbb{R} -bilinear operations

$$[\cdot, \cdot]_{\text{Leibniz}} : C^\infty(\wedge^{n-1} T^* \mathcal{M}) \times C^\infty(\wedge^{n-1} T^* \mathcal{M}) \longrightarrow C^\infty(\wedge^{n-1} T^* \mathcal{M}),$$

given by

$$[\alpha, \beta] = L_{J\alpha} \beta + (-1)^n (i(d\alpha)\eta) \beta,$$

for all $\alpha, \beta \in C^\infty(\wedge^{n-1} T^* \mathcal{M})$.

Any Leibniz algebra (\mathcal{A}, \circ) satisfies

$$f \circ (g \circ h) = (f \circ g) \circ h + g \circ (f \circ h), \quad f, g, h \in C^\infty(\mathcal{M}).$$

A morphism in the Nambu–Poisson category is a map

$$\phi : M \longrightarrow N$$

between Nambu–Poisson manifolds preserving Nambu–Poisson brackets:

$$\{f_1, \dots, f_n\}_N \circ \phi = \{f_1 \circ \phi, \dots, f_n \circ \phi\}_M.$$

The fundamental difference between the Nambu–Poisson bracket and the classical Poisson case is that for $n \geq 3$ the Nambu–Poisson polyvector η is decomposable, i.e. it has rank n at points where it does not vanish.

The Nambu–Poisson polyvectors were characterized by Takhtajan. The polyvector field η is a Nambu–Poisson tensor if and only if the natural component of η satisfy certain conditions. It was stated in [4] that the fundamental identity (2.3) is equivalent to the following algebraic and differential constraints on the Nambu–Poisson tensor η :

$$(A) \quad S_{ij} + P(S)_{ij} = 0,$$

for all multi-indices $i = \{i_1, \dots, i_n\}$ and $j = \{j_1, \dots, j_n\}$ from the set $\{1, \dots, N\}$, where

$$S_{ij} = \eta_{i_1 \dots i_n} \eta_{j_1 \dots j_n} + \eta_{j_n i_1 i_3 \dots i_n} \eta_{j_1 \dots j_{n-1} i_2} + \dots \\ + \eta_{j_n i_2 \dots i_{n-1} i_1} \eta_{j_1 \dots j_{n-1} i_n} - \eta_{j_n i_2 \dots i_n} \eta_{j_1 \dots j_{n-1} i_1};$$

(B) P is the permutation operator which interchanges the indices i_1 and j_1 of $2n$ -tensor S , and

$$\sum_{l=1}^N \left(\eta_{i_2 \dots i_n} \frac{\partial \eta_{j_1 \dots j_n}}{\partial x_l} + \eta_{j_n i_3 \dots i_n} \frac{\partial \eta_{j_1 \dots j_{n-1} i_2}}{\partial x_l} + \dots + \eta_{j_n i_2 \dots i_{n-1} l} \frac{\partial \eta_{j_1 \dots j_{n-1} i_n}}{\partial x_l} \right) \\ = \sum_{l=1}^N \eta_{j_1 j_2 \dots j_{n-1} l} \frac{\partial \eta_{j_n i_2 \dots i_n}}{\partial x_l},$$

for all $i_2, \dots, i_n, j_1, \dots, j_n = 1, \dots, N$.

It was conjectured in [2] that the equation $S_{ij} = 0$ is equivalent to the condition that n -tensor η is decomposable, recently this has been proved by Alekseevsky and author [10], Since then several other proofs appeared this surprising result (see [11, 12] and references therein).

The algebraic identity is the special feature for the Nambu–Poisson structure. It appears solely in the Nambu–Poisson geometry — there is no counterpart of this structure in the usual Poisson geometry.

The following theorem describes the local structure of the Nambu–Poisson structure of order n with $n \geq 3$.

Theorem 2.1 (Local Triviality Theorem). *Let η be a Nambu–Poisson polyvector of order $n \geq 3$. Near any point at which η does not vanish there are local coordinates x_1, \dots, x_n such that*

$$\eta = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \cdots \wedge \frac{\partial}{\partial x_n}.$$

Let $\{h_1, \dots, h_{n-1}\}$ be the set of Hamiltonian functions, and $\mathcal{X}_{h_1 \dots h_{n-1}}$ be corresponding the Hamiltonian vector field. Then the equation for the integral curves of $\mathcal{X}_{h_1 \dots h_{n-1}}$ is

$$\dot{x} = J_\eta(dh_1, \dots, dh_{n-1}),$$

Hamilton's equations of motion.

2.2 Nambu–Poisson dynamics

It is also shown in [2] that Nambu dynamics on a Nambu–Poisson phase space involves $n - 1$ so-called Nambu–Hamiltonians $H_1, \dots, H_{n-1} \in C^\infty(\mathcal{M})$ and is governed by the following equations of motion

$$\frac{df}{dt} = \{f, H_1, \dots, H_{n-1}\}, \quad \forall f \in C^\infty(\mathcal{M}).$$

A solution to the Nambu–Hamilton equations of motion produces an evolution operator U_t which by virtue of the fundamental identity preserves the Nambu bracket structure on $C^\infty(M)$.

Definition 2.1. $f \in C^\infty(\mathcal{M})$ is called an integral of motion for the system if it satisfies

$$\{f, H_1, H_2, \dots, H_{n-1}\} = 0.$$

Example. Let us illustrate how Nambu–Poisson mechanics works in practise. The example is the motion of a rigid body with a torque about the major axis introduced by Bloch and Marsden [13].

Euler's equation for the rigid body with a single torque u about its major axis is given by

$$\dot{m}_1 = a_1 m_2 m_3, \quad \dot{m}_2 = a_2 m_1 m_3, \quad \dot{m}_3 = a_3 m_1 m_2 + u,$$

where $u = -km_1 m_2$ is the feedback, $a_1 = \frac{1}{I_2} - \frac{1}{I_3}$, $a_2 = \frac{1}{I_3} - \frac{1}{I_1}$ and $a_3 = \frac{1}{I_1} - \frac{1}{I_2}$. We assume $I_1 < I_2 < I_3$.

These equations can be easily recast into generalized Nambu–Hamiltonian equations of motion

$$\frac{dm_i}{dt} = \{H_1, H_2, m_i\},$$

where the right hand side is given by

$$\{H_1, H_2, m_i\} := \frac{\partial(H_1, H_2, m_i)}{\partial(m_1, m_2, m_3)}.$$

These equations involve two Hamiltonians and these are

$$H_1 = \frac{1}{2} (a_2 m_1^2 - a_1 m_2^2), \quad H_2 = \frac{1}{2} \left(\frac{a_3 - k}{a_1} m_1^2 - m_3^2 \right).$$

When $a_1 = a_2 = a_3 = 1$ and $u_3 = 0$, these set of equations reduce to a famous Euler equation or Nahm’s equation

$$\frac{dT_i}{dt} = \epsilon_{ijk} [T_j, T_k], \quad i, j, k = 1, 2, 3,$$

where T_i s are $SU(2)$ generators.

It is well known that the system of N vortices can be described by the following system of differential equations [14]

$$\dot{z}_n = i \sum_{m \neq n}^N \frac{\Gamma_m}{z_n^* - z_m^*},$$

where $z_n = x_n + iy_n$ are complex coordinate of the centre of n -th vortex.

The equation of motion of a system of three vortices can be put in the following form [15]

$$\dot{M}_i = \epsilon_{ijk} \Gamma_i M_i (M_j - M_k), \quad i, j, k = 1, 2, 3$$

where $M_1 = |z_2 - z_3|^2$, $M_2 = |z_3 - z_1|^2$ and $M_3 = |z_1 - z_2|^2$. Again, these equations can be described by Nambu–Poisson geometry. In fact, there is a large scope to apply Nambu–Poisson formulation in fluid mechanics [16].

3 Volume preserving integrable systems

In this section we shall follow the approach of Takasaki–Takebe’s [6, 7] method of area preserving diffeomorphic (or $s\text{Diff}(2)$) KP equation. In fact, they adopted their method from self dual vacuum Einstein equation theory. In the case of self dual vacuum Einstein equation, and hyperKähler geometry also area preserving diffeomorphism appear, where the spectral variable is merely parameter. But in case of $s\text{Diff}(2)$ K.P. equation the situation is different, where Takasaki–Takebe showed that one has to treat λ as a true variable and it enters into the definition of the Poisson bracket. Our construction is closely related to Takasaki–Takebe construction, only the Kähler like two form and the associated “Darboux coordinates” is replaced by volume form and the Poisson bracket is replaced by its higher order Poisson bracket called Nambu bracket.

Suppose we consider $L = L(\lambda, p, q)$, $M = M(\lambda, p, q)$ and $N = N(\lambda, p, q)$ are some Laurent series in λ with coefficients are functions of p and q .

Definition 3.1. The volume preserving integrable hierarchy is defined by

$$\frac{\partial L}{\partial t_n} = \{B_{1n}, B_{2n}, L\}, \quad (3.1)$$

$$\frac{\partial M}{\partial t_n} = \{B_{1n}, B_{2n}, M\}, \quad (3.2)$$

$$\frac{\partial N}{\partial t_n} = \{B_{1n}, B_{2n}, N\}, \quad (3.3)$$

and

$$\{L, M, N\} = 1, \quad (3.4)$$

where $B_{1n} := (L^n)_{n \geq 0}$ and $B_{2n} := (M^n)_{n \geq 0}$. The first three equations are hierarchy equations, and these generate the three flows of the system and the last one shows the volume preservation condition. The Nambu–Poisson bracket in 3D “phase space” (λ, p, q) is given by

$$\{A(\lambda, p, q), B(\lambda, p, q), C(\lambda, p, q)\} = \frac{\partial A}{\partial \lambda} \left(\frac{\partial B}{\partial p} \frac{\partial C}{\partial q} - \frac{\partial B}{\partial q} \frac{\partial C}{\partial p} \right) + \text{cyclic terms}.$$

Let us now compare our case with the area preserving KP hierarchy. The sdiff(2) KP hierarchy is given by

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{L}\}, \quad \frac{\partial \mathcal{K}}{\partial t_n} = \{\mathcal{B}_n, \mathcal{K}\}, \quad \{\mathcal{L}, \mathcal{K}\} = 1,$$

where \mathcal{L} is a Laurent series in an indeterminate λ of the form

$$\mathcal{L} = \lambda + \sum_{n=1}^{\infty} u_{n+1}(t) \lambda^{-n},$$

$\mathcal{B}_n = (\mathcal{L})_{\geq 0}$. The function \mathcal{K} is called Orlov function and it is defined by

$$\mathcal{K} = \sum_{n=1}^{\infty} n t_n \mathcal{L}^{n-1} + x + \sum_{i=1}^{\infty} v_i \mathcal{L}^{-i-1},$$

where $t_1 = x$.

Remark 3.1. Our hierarchy has a structure of volume preserving KP hierarchy and instead of one Orlov function \mathcal{K} , we need two Orlov functions M and N .

Any two equations of the hierarchy commute

$$\partial_{t_m} \partial_{t_l} = \partial_{t_l} \partial_{t_m},$$

hence we obtain

Proposition 3.1. *The Lax equation for L , M , N are equivalent to the following equations:*

$$\begin{aligned} & \left\{ \frac{\partial B_{1n}}{\partial t_m}, B_{2n} \right\} + \left\{ B_{1n}, \frac{\partial B_{2n}}{\partial t_m} \right\} + \left\{ \hat{B}_1, B_{2m} \right\} \\ & - \left\{ \frac{\partial B_{1m}}{\partial t_n}, B_{2m} \right\} - \left\{ B_{1m}, \frac{\partial B_{2m}}{\partial t_n} \right\} + \left\{ B_{1m}, \hat{H}_2 \right\} = 0, \end{aligned}$$

where

$$\hat{H}_1 = \{B_{1n}, B_{2n}, B_{1m}\}, \quad \hat{H}_2 = \{B_{1n}, B_{2n}, B_{2m}\}.$$

Proof. Result follows from the compatibility conditions of hierarchy equations and the fundamental identity. ■

Remark 3.2. In the case of sDiff(2) hierarchy the above expression boils down to zero curvature equation.

Let $\Omega^{(3)}$ be a three form given by

Definition 3.2.

$$\Omega^{(3)} := \sum_{n=1}^{\infty} dB_{1n} \wedge dB_{2n} \wedge dt_n = d\lambda \wedge dp \wedge dq + \sum_{n=2}^{\infty} dB_{1n} \wedge dB_{2n} \wedge dt_n. \quad (3.5)$$

From the definition it is clear Ω is closed 3 form. In fact sDiff(3) structure is clearly exhibited from this structure and the theory is integrable in the sense of nonlinear graviton construction [14]. This is a generalization of nonlinear graviton construction.

Theorem 3.1. *The volume preserving hierarchy is equivalent to the exterior differential equation*

$$\Omega^{(3)} = dL \wedge dM \wedge dN. \quad (3.6)$$

Proof. We have seen that Ω can be written in two ways. Expanding both sides of the exterior differential equation as linear combinations of $d\lambda \wedge dp \wedge dq$, $d\lambda \wedge dp \wedge dt_n$, $d\lambda \wedge dq \wedge dt_n$ and $dp \wedge dq \wedge dt_n$.

When we pick up the coefficients of $d\lambda \wedge dp \wedge dq$, we obtain the volume preserving condition

$$\{L, M, N\} = 1.$$

When we equate the other coefficients, viz. $d\lambda \wedge dp \wedge dt_n$, $d\lambda \wedge dq \wedge dt_n$ and $dp \wedge dq \wedge dt_n$ we obtain the following identities:

$$\frac{\partial(B_{1n}, B_{2n})}{\partial(\lambda, p)} = \frac{\partial(L, M, N)}{\partial(\lambda, p, t_n)}, \quad \frac{\partial(B_{1n}, B_{2n})}{\partial(\lambda, q)} = \frac{\partial(L, M, N)}{\partial(\lambda, q, t_n)},$$

and

$$\frac{\partial(B_{1n}, B_{2n})}{\partial(p, q)} = \frac{\partial(L, M, N)}{\partial(p, q, t_n)}$$

respectively. We multiply the above three equations by $\frac{\partial L}{\partial q}$, $\frac{\partial L}{\partial p}$ and $\frac{\partial L}{\partial \lambda}$ respectively. If we add first and third equations and subtract the second one from them, then after using the volume preserving identity we obtain

$$\frac{\partial L}{\partial t_n} = \{B_{1n}, B_{2n}, L\}.$$

Similarly we can obtain the other equations also, in that case we will multiply the equations (3.1), (3.2) and (3.3) by $\frac{\partial M}{\partial q}$, $\frac{\partial M}{\partial p}$ and $\frac{\partial M}{\partial \lambda}$ respectively. ■

Equation (3.5) and (3.6), we obtain

$$d \left(MdL \wedge dN + \sum_{n=1}^{\infty} B_{1n} dB_{2n} \wedge dt_n \right) = 0.$$

This implies the existence of one form Q such that

$$dQ = Md(LdN) + \sum_{n=1}^{\infty} B_{1n} d(B_{2n} dt_n).$$

This is an analogue of “Krichever potential” in the volume preserving case. Hence we can say from (3.4)

$$M = \frac{\partial Q}{\partial(LdN)} \Big|_{B_{2n}, t_n \text{ fixed}}, \quad B_{1n} = \frac{\partial Q}{\partial(B_{2n} dt_n)} \Big|_{L, N, B_{2m}, t_m (m \neq n) \text{ fixed}}.$$

4 Application to multidimensional integrable systems and Riemann–Hilbert problem

We already stated that our situation is quite similar to nonlinear graviton construction of Penrose [17] for the self dual Einstein equation. This is a generalization of nonlinear graviton constructions.

To the geometer self dual gravity is nothing but Ricci flat Kähler geometry and it is characterized by the underlying symmetry groups $\text{sDiff}(2)$, this are called area preserving diffeomorphism group on surfaces. These are the natural generalization of the groups $\text{Diff}(S^1)$, diffeomorphism of circle.

It is well known how the area preserving diffeomorphism group appears in the self dual gravity equation. Let us give a very rapid description of this.

Let us start from a complexified metric of the following form

$$ds^2 = \det \begin{pmatrix} e^{11} & e^{12} \\ e^{21} & e^{22} \end{pmatrix} = e^{11} e^{22} - e^{12} e^{21},$$

where e^{ij} are independent one forms. Ricci flatness condition boils down to the closedness of

$$d\Omega^{kl} = 0 \tag{4.1}$$

of the exterior 2-forms

$$\Omega^{kl} = \frac{1}{2} J_{ij} e^{ik} \wedge e^{jl},$$

where J is the normalized symplectic form

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then above system of two forms can be recast to

$$\Omega(\lambda) = \frac{1}{2} J_{ij} (e^{i1} + e^{i2} \lambda) \wedge (e^{j1} + e^{j2} \lambda).$$

This satisfies

$$\Omega(\lambda) \wedge \Omega(\lambda) = 0, \quad d\Omega(\lambda) = 0, \quad (4.2)$$

where d stands for total differentiation. These suggest us to introduce a pair of Darboux coordinates

$$\Omega(\lambda) = dP \wedge dQ$$

and these are the sections of the twistor fibration

$$\pi : \mathcal{T} \longrightarrow CP^1,$$

where \mathcal{T} is the curved twistor space introduced by Penrose. Basically each fibre is endowed with a symplectic form and as the base point moves this also deforms and here comes the area preservation.

Two pairs of Darboux coordinates are related by

$$f(\lambda, P(\lambda), Q(\lambda)) = P', \quad g(\lambda, P(\lambda), Q(\lambda)) = Q'$$

and f and g satisfy $\{f, g\} = 1$. The pair (f, g) is called twistor data. Locally f and g (after twisting with λ) yield patching function. Ricci flat Kähler metric is locally encoded in this data. This set up is nothing but the Riemann–Hilbert problem in area preserving diffeomorphism case.

The novelty of this approach is that this twistor construction will work in the higher dimensions too, when there is no twistor projection. The most important example is the electro-vacuum equation, volume preserving diffeomorphism groups in three dimension play a vital role here. This model was first introduced by Flaherty [18] and later Takasaki [2] showed how this works explicitly.

4.1 Gindikin's bundle of forms

We already stated that anti-self dual vacuum equations govern the behaviour of complex 4-metrics of signature $(+, +, -, -)$ whose Ricci curvature is zero and whose Weyl curvature is self dual. These two curvatures are independent of change of coordinates, so in one particular of the equations these metric becomes automatically Kähler and can be

expressed in terms of a single scalar function Ω , the Kähler potential. Then curvature conditions will lead you to Ist Plebanski's Heavenly equation

$$\frac{\partial^2 \tilde{\Omega}}{\partial x \partial \tilde{x}} \frac{\partial^2 \tilde{\Omega}}{\partial y \partial \tilde{y}} - \frac{\partial^2 \tilde{\Omega}}{\partial x \partial \tilde{y}} \frac{\partial^2 \tilde{\Omega}}{\partial y \partial \tilde{x}} = 1,$$

or using the Poisson bracket (w.r.t x and y)

$$\{\tilde{\Omega}_{\tilde{x}}, \tilde{\Omega}_{\tilde{y}}\} = 1,$$

and the corresponding anti-self-dual Ricci flat metric is

$$g(\tilde{\Omega}) = \frac{\partial^2 \tilde{\Omega}}{\partial x^i \partial \tilde{x}^j} dx^i d\tilde{x}^j, \quad \tilde{x}^i = \tilde{x}, \tilde{y}, \quad x^j = x, y.$$

This system is completely integrable and it is an example of a multidimensional integrable system.

Let Ω be the 2-form, given by

$$\begin{aligned} \Omega(x, \tilde{x}, y, \tilde{y}) &= dx \wedge dy \\ &+ \lambda(\tilde{\Omega}_{x\tilde{x}} dx \wedge d\tilde{x} + \tilde{\Omega}_{x\tilde{y}} dx \wedge d\tilde{y} + \tilde{\Omega}_{y\tilde{x}} dy \wedge d\tilde{x} + \tilde{\Omega}_{y\tilde{y}} dy \wedge d\tilde{y}) + \lambda^2 d\tilde{x} \wedge d\tilde{y}. \end{aligned}$$

Lemma 4.1.

- 1) $d\Omega = 0$,
- 2) $\Omega \wedge \Omega = 0$.

Proof. Since Ω satisfies Plebanski's Heavenly Ist equation, hence 2) is true. ■

A number of multidimensional integrable systems can be written in terms of a 2-form Ω which satisfies the equations (4.1).

Example. The dispersionless KP hierarchy has a Lax representation with respect to a series of independent ("time") variables $t = (t_1, t_2, \dots)$

$$\frac{\partial \mathcal{L}}{\partial t_n} = \{B_n, \mathcal{L}\},$$

where $B_n := (\mathcal{L}^n)_{\geq 0}$, $n = 1, 2, \dots$, \mathcal{L} is a Laurent series in an indeterminant λ of the form

$$\mathcal{L} = \lambda + \sum_{n=1}^{\infty} u_{n+1}(t) \lambda^{-n},$$

$\{, \}$ is a Poisson bracket in 2D phase space with respect to (λ, x) .

Let us consider

$$\Omega = d\lambda \wedge dx + \sum_{n=2}^{\infty} dB_n \wedge dt_n$$

then $\Omega \wedge \Omega$ is equivalent to the zero curvature condition

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + \{B_n, B_m\} = 0.$$

This is an alternative form of dispersionless KP hierarchy.

All these systems are related to area preserving diffeomorphism group $\text{sDiff}(2)$.

Proposition 4.1. *The 3-form*

$$\Omega^{(3)} = d\lambda \wedge dp \wedge dq + \sum_{n=2}^{\infty} dB_{1n} \wedge dB_{2n} \wedge dt_n,$$

$$d\Omega^{(3)} = 0, \quad \Omega^{(3)} \wedge \Omega^{(3)} = 0.$$

Proof. This is equivalent to Proposition 3.1, hence it is satisfied.

The Gindikin's method [19] of pencil of 2-forms is the most effective way to study these systems. Consider the following system of 1st order equation depending on a parameter $\tau = (\tau_1, \tau_2) \in \mathbb{C}^2$

$$\begin{aligned} e^1(\tau) &= e^{11}\tau_1^k + \cdots + e^{1k}\tau_2^k, \\ e^2(\tau) &= e^{21}\tau_1^k + \cdots + e^{2k}\tau_2^k, \\ &\vdots \\ e^{2l}(\tau) &= e^{2l1}\tau_1^k + \cdots + e^{2lk}\tau_2^k, \end{aligned}$$

where e^{ij} are 1-forms. Let $\Omega^k(\tau)$ be the bundle of 2-forms

$$\Omega^k(\tau) = e^1(\tau) \wedge e^2(\tau) + \cdots + e^{2l-1}(\tau) \wedge e^{2l}(\tau)$$

satisfying the conditions

$$\left(\Omega^k\right)^{l+1}=0, \quad\left(\Omega^k\right)^l \neq 0, \quad d \Omega^k=0 .$$

The bundle of forms actually encodes the integrability of the original system. In the special case $l = 1$, $k = 2$, we recover the Ricci flat metric

$$g = e^{11}e^{22} - e^{12}e^{21}.$$

4.2 Higher dimensional analogue of Gindikin's Pencil

Let us consider the following system of Ist order equations depending on parameter $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbb{C}^3$,

$$\begin{aligned} e^1(\tau) &= e^{11}\tau_1 + e^{12}\tau_2 + e^{13}\tau_3, \\ e^2(\tau) &= e^{21}\tau_1 + e^{22}\tau_2 + e^{23}\tau_3, \\ e^3(\tau) &= e^{31}\tau_1 + e^{32}\tau_2 + e^{33}\tau_3. \end{aligned}$$

Just like the previous situation these bundle of forms then encodes the integrability of the original system. The metric defined here is the higher order analogue of Ricci metric in two form case.

This metric is given by,

$$q = e^{11}e^{22}e^{33} - e^{11}e^{32}e^{23} + e^{12}e^{31}e^{23} - e^{12}e^{21}e^{33} + e^{13}e^{21}e^{32} - e^{13}e^{31}e^{22}.$$

Remark 4.1. Since 3-form $\Omega^{(3)}$ satisfies

$$d\Omega^{(3)} = 0, \quad \Omega^{(3)} \wedge \Omega^{(3)} = 0,$$

so it denotes the volume preserving multidimensional integrable systems. These systems can be described by Gindikin's bundle of multi forms, higher dimensional analogue of nonlinear graviton.

4.3 Twistor description of volume preserving multidimensional integrable systems

The natural question would be to find out the analogous Riemann–Hilbert problem in the volume preserving case. Our situation is very similar to electro-vacuum equation. Let us consider two sets of solutions of hierarchy (L, M, N) and $(\hat{L}, \hat{M}, \hat{N})$ with different analysity. Then there exist an invertible functional relation between these two sets of functions such that it satisfies

$$\hat{L} = f_1(L, M, N), \quad \hat{M} = f_2(L, M, N), \quad \hat{N} = f_3(L, M, N),$$

where $f_1 = f_1(\lambda, p, q)$, $f_2 = f_2(\lambda, p, q)$ and $f_3 = f_3(\lambda, p, q)$ are arbitrary holomorphic functions defined in a neighbourhood of $\lambda = \infty$ except at $\lambda = \infty$.

We assume f_1, f_2, f_3 satisfy the canonical Nambu Poisson relation

$$\{f_1, f_2, f_3\} = 1.$$

This is a kind of Riemann–Hilbert problem related to three dimensional diffeomorphisms. In this case $\text{sDiff}(3)$ symmetries is clear, in fact $\text{sDiff}(3)$ group acts on (f_1, f_2, f_3) , we can lift this action on (L, M, N) and $(\hat{L}, \hat{M}, \hat{N})$ via Riemann–Hilbert factorization.

4.4 Application to hydrodynamic type systems

There are certain kind of integrable systems, called hydrodynamic type, naturally arise in gas dynamics, hydrodynamics, chemical kinematics and may other situations, these are given by

$$u_t^i = v_j^i(u) u_x^j,$$

where $v_j^i(u)$ is an arbitrary $N \times N$ matrix function of

$$u = (u^1, \dots, u^N), \quad u^i = u^i(x, t), \quad i = 1, \dots, N.$$

The Hamiltonian systems of hydrodynamic type systems considered above have the form

$$u_t^i = \{u^i, H\},$$

where $H = \int h(u) dx$, is a functional of hydrodynamic type. The Poisson bracket of these systems has the form

$$\{u^i(x), u^j(y)\} = g^{ij}(u) \delta_x(x - y) + b_k^{ij}(u) u_x^k \delta(x - y),$$

called Dubrovin–Novikov type Poisson bracket. Dubrovin–Novikov showed if the metric is non degenerate i.e. $\det [g^{ij}] \neq 0$ then the above bracket yields a Poisson bracket provided $g^{ij}(u)$ is a metric of zero Riemannian Curvature.

A large class of them can be described by the Lax form,

$$\psi_x = zA\psi, \quad \psi_t = zB\psi.$$

From the compatibility condition we obtain,

$$A_t = B_x, \quad AB = BA.$$

These set of equations can be easily recasted to

$$d\Omega = 0, \quad \Omega \wedge \Omega = 0,$$

where Ω is a closed one form.

Hence has also a twistorial description. The upshot of this section is that Gindikin pencil of forms can be applied to large number of classes integrable systems, including the volume preserving integrable systems we propose here.

5 Conclusion

In this paper we have shown that there are large class of integrable systems can be obtained from Nambu–Poisson mechanics. They belong to the same family of self dual Einstein or dispersionless KP type equations. In some sense these integrable systems are higher dimensional generalization of self dual Einstein equation. Hence these systems are describable via Gindikin’s bundle of forms, or twistor method. Our integrable systems are volume preserving. We hope that eventually such construction will find their use in the construction of volume preserving KP and Toda type equation.

We have also discussed in this paper that Nambu–Poisson manifold is an useful tool to study volume preserving integrable systems. Recently in membrane theory physicists [20] have found M-algebra from M-brane, these are related to Nambu–Poisson mechanics.

There are certain problems we have not discussed in this paper, viz. the quantization of these volume preserving generalized multidimensional integrable systems. Presumably, the method of star product quantization [21, 22, 23] would be the best way quantize these systems and instead of binary star product we need triple star product [24].

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References

- [1] Guha P, Volume Preserving Integrable Systems and Nambu–Poisson Manifolds, *Class. Quant. Gravity* **14** (1997), L37–L43.

- [2] Takasaki K, Volume Preserving Diffeomorphisms in Integrable Deformation of Self-Dual Gravity, *Phys. Lett.* **B285** (1992), 187–190.
- [3] Nambu Y, Generalized Hamiltonian Dynamics, *Phys. Rev.* **D7** (1973), 2405.
- [4] Takhtajan L A, On Foundation of the Generalized Nambu Mechanics, *Comm. Math. Phys.* **160** (1994), 295.
- [5] Chatterjee R and Takhtajan L A, Aspects of Classical and Quantum Nambu Mechanics, *Lett. Math. Phys.* **37** (1996), 475.
- [6] Takasaki K and Takebe T, SDiff(2) KP hierarchy, *Int. J. Mod. Phys.* **7A** (1992), 889.
- [7] Takasaki K and Takebe T, Integrable Hierarchies and Dispersionless Limits, preprint, UTMS 94-35.
- [8] Michor P W and Vaisman I, A Note on n -ary Poisson Brackets, in Proceedings of the 19th Winter School “Geometry and Physics” (Srn, 1999), *Rend. Circ. Mat. Palermo* (2) Suppl., N 63 (2000), 165–172.
- [9] Ibanez R, de Leon M, Marrero J C and Padron E, Leibniz Algebroid Associated with a Nambu–Poisson Structure, *J. Phys.* **A32** (1999), 8129–8144.
- [10] Alekseevsky D V and Guha P, On Decomposibility of Nambu Tensor, *Acta Math. Com. Univ.* **65** (1996), 1.
- [11] Marmo G, Vilasi G. and Vinogradov A M, The Local Structure of n -Poisson and n -Jacobi Manifolds, *J. Geom. Phys.* **25** (1998), 141.
- [12] Vaisman I, A Survey on Nambu–Poisson Brackets, *Acta Math. Univ. Comenian. (N.S.)* **68** (1999), 213–241.
- [13] Bloch A M and Marsden J E, Stabilization of Rigid Body Dynamics by the Energy–Casimir Method, *Systems and Control Letters* **14** (1990), 341.
- [14] Aref H, *Ann. Rev. Fluid Mech.* **15** (1983), 345.
- [15] Makhaldiani, The System of Three Vortexes of Two Dimensional Ideal Hydrodynamics as a New Example of the (Integrable) Nambu–Poisson Mechanics, solv-int/9804002.
- [16] Guha P, in preparation.
- [17] Penrose R P, Nonlinear Gravitons and Curved Twistor Theory, *Gen. Rel. Grav.* **31** (1976), 7.
- [18] Flaherty E J, The Nonlinear Graviton in Interaction with Photon, *Gen. Rel. Grav.* **9** (1978), 961.
- [19] Gindikin S G, Generalized Conformal Structures, in *Twistors in Mathematics and Physics*, Editors: Bailey T N and Baston R J, *LMS Lecture Note Series*, CUP - Cambridge, **156**, 1990.
- [20] Hoppe J, On M-Algebras, the Quantization of Nambu-Mechanics and Volume Preserving Diffeomorphisms, hep-th 9602020, ETH-TH/95-33.
- [21] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D, Deformation Theory and Quantization: I. Deformations of Symplectic Structures, *Ann. Phys.* **110** (1978), 67.
- [22] Kontsevich M, Deformation Quantization of Poisson Manifolds, Preprint IHES/M/97/72, q-alg/9709040.

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- [23] Strachan I A B, A Geometry for Multidimensional Integrable Systems, *J. Geom. Phys.* **21** (1997), 255.
- [24] Dito G, Flato M, Sternheimer D and Takhtajan L, Deformation Quantization and Nambu Mechanics, *Comm. Math. Phys.* **183** (1997), 1.